The Ensemble Transform Particle Filter (ETPF)

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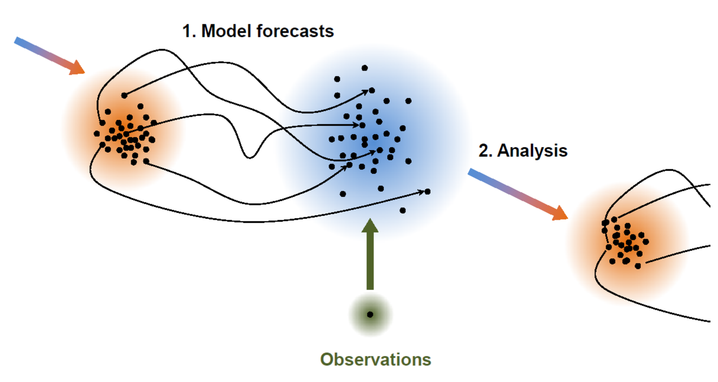


Figure from *Université catholique de Louvain*
http://www.elic.ucl.ac.be/repomodx/elic/index.php?id=393

Motivation

The *Ensemble Transform Kalman Filter* (ETKF)[1] analysis step relies on the Gaussian assumption for prior density and observation. In contrast, the *Particle Filter* (PF)[2] performs an exact Bayesian analysis, but is highly unstable.

Approach

We replace the ETKF analysis step with a *second-order exact deterministic update*, derived from the PF.
Let $\mathbf{X}_f$ be the matrix containing the forecast (prior) ensemble, and $\mathbf{X}_f'$ are its perturbations (deviations from ensemble mean).

ETKF Analysis Step

Prior statistics

Forecast mean and covariance

$$\overline{\mathbf{x}}_f = \frac{1}{m}\sum_{i=1}^{m}\mathbf{x}_f^i = \frac{1}{m}\mathbf{X}_f\mathbf{1} \qquad \mathbf{P}_f = \frac{1}{m-1}\mathbf{X}_f'(\mathbf{X}_f')^T$$

Targeted analysis statistics

Analysis mean and covariance from Kalman filter

$$\overline{\mathbf{x}}_a^{\mathrm{KF}} = \overline{\mathbf{x}}_f + \mathbf{K}\left(\mathbf{y} - \overline{\mathcal{H}(\mathbf{x}_f)}\right)$$
$$\mathbf{P}_a^{\mathrm{KF}} = (\mathbf{I}_n - \mathbf{K}\mathbf{H})\mathbf{P}_f$$

Ensemble transformation

Deterministic update of the complete ensemble

$$\mathbf{X}_a' = \sqrt{m-1}\,\mathbf{X}_f'\mathbf{T}.$$

$$\mathbf{T}\mathbf{T}^T = \left[(m-1)\mathbf{I}_k + (\mathbf{H}\mathbf{X}_f')^T\mathbf{R}^{-1}(\mathbf{H}\mathbf{X}_f')\right]^{-1}$$

Discussion

- Very stable, no stochastic error as in the classical EnKF
- State-of-the-art filter with many applications & extensions
- Implicit assumption of Gaussianity → not exact in nonlinear case

PF Analysis Step

Prior ensemble

Weighting the prior ensemble with the observational likelihood

$$w^i = \frac{p(\mathbf{y}|\mathbf{x}_f^i)}{\sum_{i=1}^{m} p(\mathbf{y}|\mathbf{x}_f^i)}$$

Analysis ensemble

Only weights are modified by the standard PF

$$\overline{\mathbf{x}}_a^{\mathrm{PF}} = \sum_{i=1}^{m} w^i\mathbf{x}_f^i$$
$$\mathbf{P}_a^{\mathrm{PF}} = \sum_{i=1}^{m} w^i\left(\mathbf{x}_f^i - \overline{\mathbf{x}}_a^{\mathrm{PF}}\right)\left(\mathbf{x}_f^i - \overline{\mathbf{x}}_a^{\mathrm{PF}}\right)^T$$

Discussion

- Non-parametric filter, exact analysis
- Severe tendency to filter collapse (divergence)

Other approaches

- Resampling of the analysis ensemble → not sufficient
- Proposal density[3] during forecast step → more promising

Ensemble Transform Particle Filter (ETPF) Analysis Step

(1) Exact 2nd order statistics ($\overline{\mathbf{x}}$, $\mathbf{P}_a$)

Based on prior ensemble and PF weights $\mathbf{w}=(w_i)$

$$w^i \propto p(\mathbf{y}|\mathbf{x}_f^i) \qquad \mathbf{W} \equiv \mathrm{diag}(\mathbf{w})$$

(2) Deterministic update

Generate analysis ensemble with this mean & covariance by transformation of the prior ensemble (with matrix **T**)

$$\overline{\mathbf{x}}_a^{\mathrm{PF}} = \overline{\mathbf{x}}_f + \mathbf{X}_f'\mathbf{w}$$
$$\mathbf{P}_a^{\mathrm{PF}} = \mathbf{X}_f'\left(\mathbf{W} - \mathbf{w}\mathbf{w}^T\right)(\mathbf{X}_f')^T$$

$$\mathbf{T}^{\mathrm{PF}} = \left(\mathbf{W} - \mathbf{w}\mathbf{w}^T\right)^{1/2}$$

(3) Random ensemble transformation

Additional transformation in ensemble space[4]
Conserves 2nd order statistics *exactly*
Produces a more Gaussian distribution → stabilizes filter

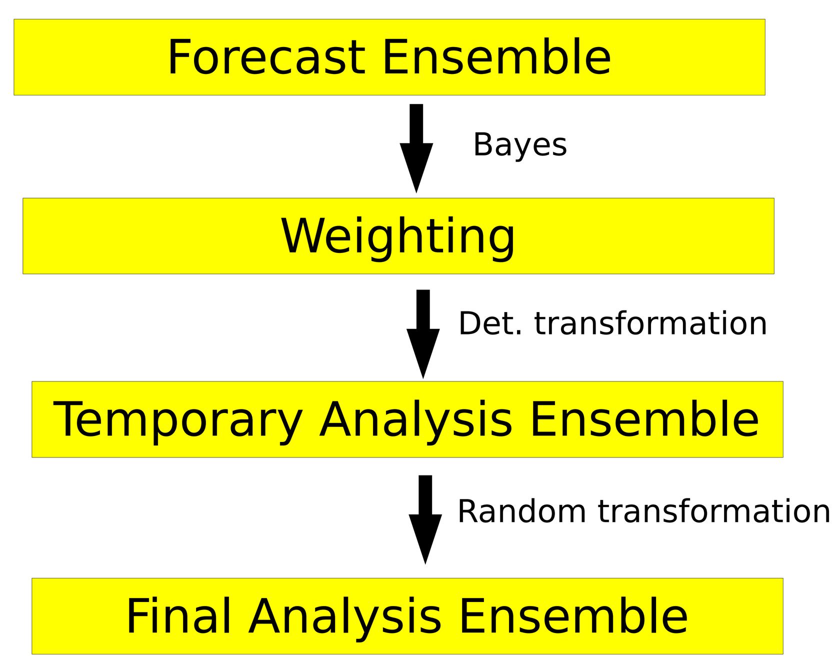
$\mathbf{\Lambda}$ — *Random matrix from 2nd order exact sampling* [4]

$$\mathbf{X}_a'^{\,\mathrm{PF}} = \sqrt{m-1}\,\mathbf{X}_f'\mathbf{T}^{\mathrm{PF}}\mathbf{\Lambda}$$

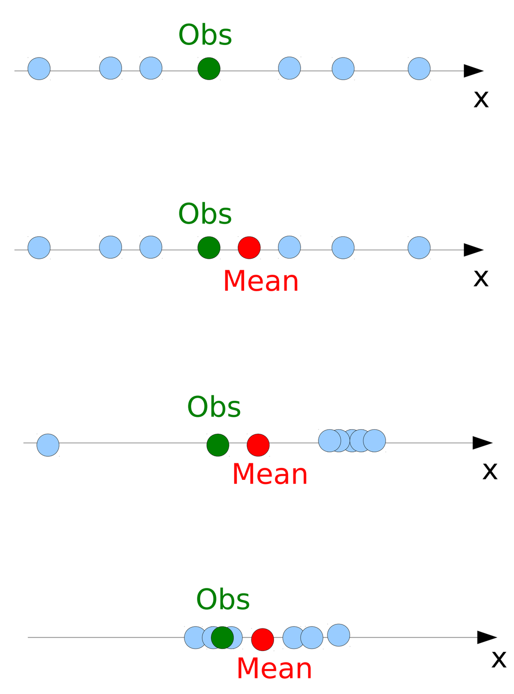
(4) Localization

A local analysis can be performed as in the LETKF

Transformations in the ETPF:



Schematic visualization on a scalar example (m=6):



Application to State Estimation in the Lorenz 63 System

Toy model

Low-dimensional, but highly chaotic:
Can the filter reconstruct the model truth using only partial, imperfect observations?

Setup

→N_ens = 100
→ens_init = climatology
→obs_density = dt*25 (dt=0.01)
→obs_which = x,z
→obs_error =2

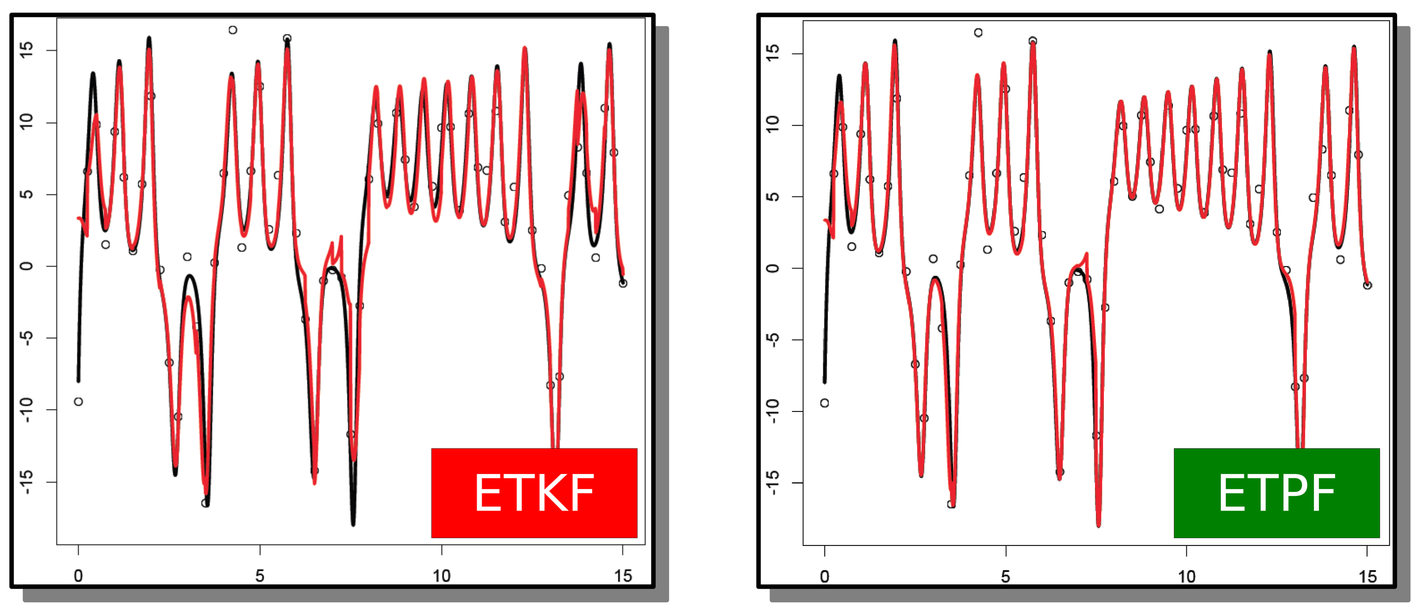


Fig.1: x-component of L63, analysis (**red**) versus truth (**black**) for both filters

Comparison

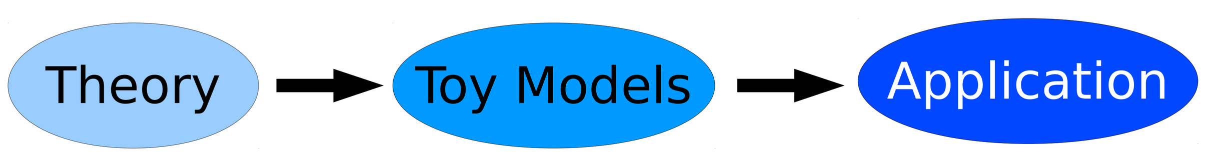
- Small error compared to ETKF
- Stable trajectory, no divergence
- Statistically consistent
- No underdispersion

	ETKF	ETPF
RMSE (of ens. mean)	1.67	0.69
CRPS (ensemble)	1.17	0.43
% of truth in 95% confidence interval	x: 80.5%, y: 81.1%, z: 79.5%	x: 95.3%, y: 95.6%, z: 93.8%

Summary and Outlook

Summary

↪Derivation of a 2nd order exact deterministic filter
↪Motivating results with toy models
↪Localization possible as in the LETKF
↪Publication ongoing



Future Work

- Further investigation of localization
- Behavior in higher-dimensional models
- Application to parameter estimation
- …

References

[1] Hunt, B. R et.al., 2007: Efficient data assimilation for spatiotemporal chaos: A local ensemble transform Kalman filter. *Physica D*, 230, 112–126.
[2] Gordon, N. et.al., 1993: Novel approach to nonlinear/non-gaussian Bayesian state estimation. *IEE Proceedings F,* 140 (2), 107–113.
[3] van Leeuwen, P. J. and M. Ades, 2013: Efficient fully non- linear data assimilation for geophysical fluid dynamics. *Computers & Geosciences,* 55, 16–27.
[4] Nerger, L., T. Janjíc, J. Schröter, and W. Hiller, 2012: A unification of ensemble square root Kalman filters. *Monthly Weather Review,* 140, 2335–2345.

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