3D diffusion in steep terrain: testing and stability of horizontally explicit, vertically implicit discretizations

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Michael Baldauf
Deutscher Wetterdienst, Offenbach, Germany

many thanks for discussions with Slavko Brdar and Günther Zängl (DWD)

3D diffusion in general coordinate systems

scalar diffusion equation

$$\rho \frac{\partial s}{\partial t} = -\nabla_j H^j = -\frac{\partial}{\partial x^j} H^j - \Gamma^j_{jk} H^k$$

vectorial diffusion equation

$$\rho \frac{\partial v^i}{\partial t} = -\nabla_j T^{ij} = -\frac{\partial}{\partial x^j} T^{ij} - \Gamma^i_{jk} T^{kj} - \Gamma^j_{jk} T^{ik}$$

with gradient expressions for scalar diffusion flux vector

$$H^i = -\rho K_s \, g^{ij} \nabla_j s$$

and momentum flux (stress) tensor

$$T^{ij} = -\rho K_m \left(g^{il} \nabla_l v^j + g^{jl} \nabla_l v^i \right) - \rho K_d g^{ij} \nabla_l v^l$$

These formulae cannot be used directly since many meteorological models use *spherical* (i.e. *non-terrain-following*) and *normalized* base vectors →

Diffusion in spherical + terrain-following coordinates $\lambda, \phi, \zeta(\lambda,\phi,r=R+z)$

scalar flux divergence:

terrain following coordinates — vertical

$$\rho \frac{\partial s}{\partial t} = -\frac{1}{r\cos\phi}\frac{\partial H^{*1}}{\partial \lambda} - \frac{1}{r\cos\phi}\frac{\partial \zeta}{\partial \lambda}\frac{\partial H^{*1}}{\partial \zeta} - \frac{1}{r}\frac{\partial H^{*2}}{\partial \phi} - \frac{1}{r}\frac{\partial \zeta}{\partial \phi}\frac{\partial H^{*2}}{\partial \zeta} - \frac{\partial \zeta}{\partial z}\frac{\partial H^{*3}}{\partial \zeta}$$

$$-\frac{2}{r}H^{*3} + \frac{\tan\phi}{r}H^{*2}$$

earth curvature

horizontal (cartesian)

diffusion flux vector of scalar s (physical components):

$$H^{*1} = -\rho K_s \frac{1}{r\cos\phi}\left(\frac{\partial s}{\partial \lambda} + \frac{\partial \zeta}{\partial \lambda}\frac{\partial s}{\partial \zeta}\right)$$

$$H^{*2} = -\rho K_s \frac{1}{r}\left(\frac{\partial s}{\partial \phi} + \frac{\partial \zeta}{\partial \phi}\frac{\partial s}{\partial \zeta}\right)$$

$$H^{*3} = -\rho K_s \frac{\partial \zeta}{\partial z}\frac{\partial s}{\partial \zeta}$$

analogous:
‚vectorial' diffusion of *u*, *v*, *w*

Baldauf (2005), COSMO-Newsl. Nr. 5

Basic discretization strategy for 3D diffusion:

(here: the diffusion coefficients themselves are assumed as given)

- spatially: 2nd order centered finite differences (here: staggered C-grid)
- temporally: horizontally explicit - vertically implicit (HE-VI)
  analogous to the treatment of the Euler solver,
  only *tridiagonal* solvers are 'allowed'

HE-VI:

\+ good efficiency

\+ good scalability ← only horizontal nearest neighb. data exchange

\- **metric terms in steep terrain can become unstable**
In fact , 3D diffusion in older versions of COSMO suffer from this problem

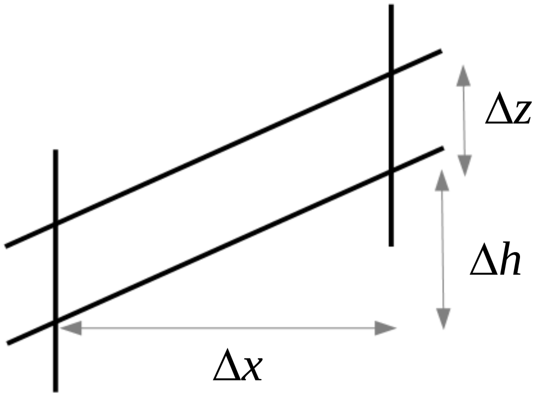
In the following:

- **how can numerical stability be improved**
- **idealized tests for validation**

Numerical stability

Linear von-Neumann stability analysis of 2D (i.e. x-z-diffusion) in tilted terrain

consider diffusion with constant diffusion coeff. K and constant steepness



$$\frac{\Delta h}{\Delta z} = m \cdot M_z$$

<u>dimensionless variables:</u>

diffusion-Courant number

$$C_{diff} := K \frac{\Delta t}{\Delta x^2}$$

steepness

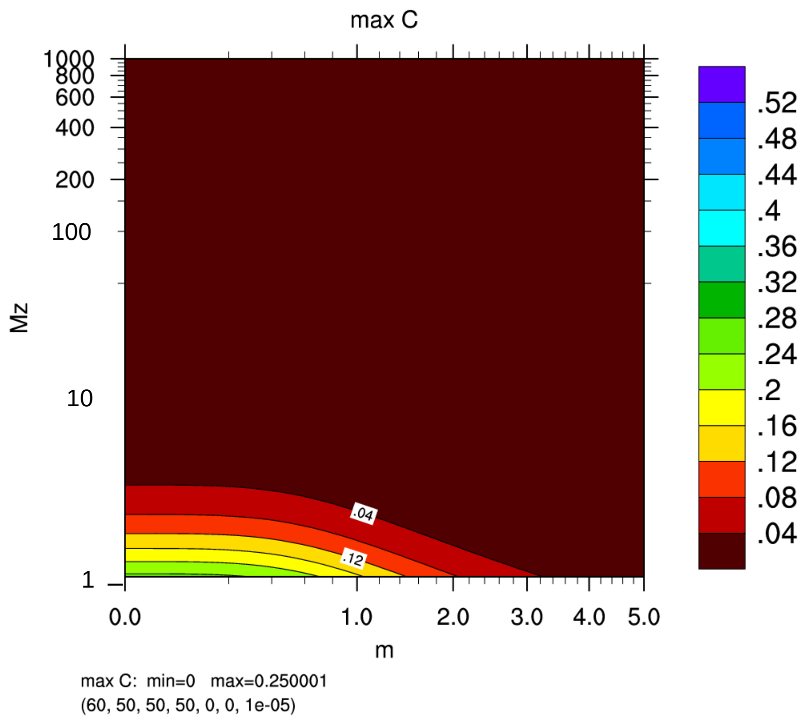
$$m := \left. \frac{\partial z}{\partial x} \right|_{\zeta} = \frac{\Delta h}{\Delta x}$$

grid anisotropy

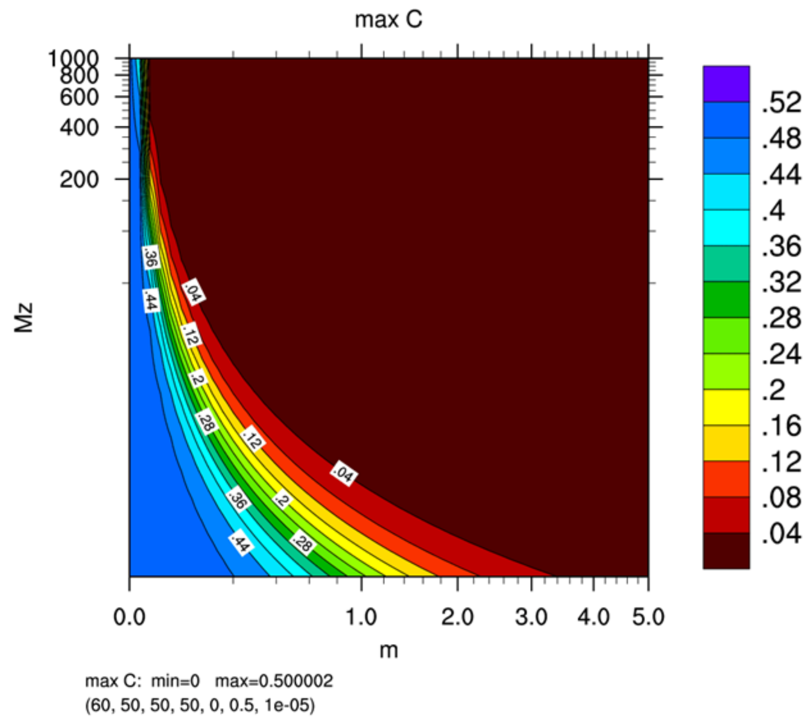
$$M_z := \frac{\Delta x}{\Delta z}$$

Stability of 3D scalar diffusion in tilted terrain: max C_{Diff}

purely explicit



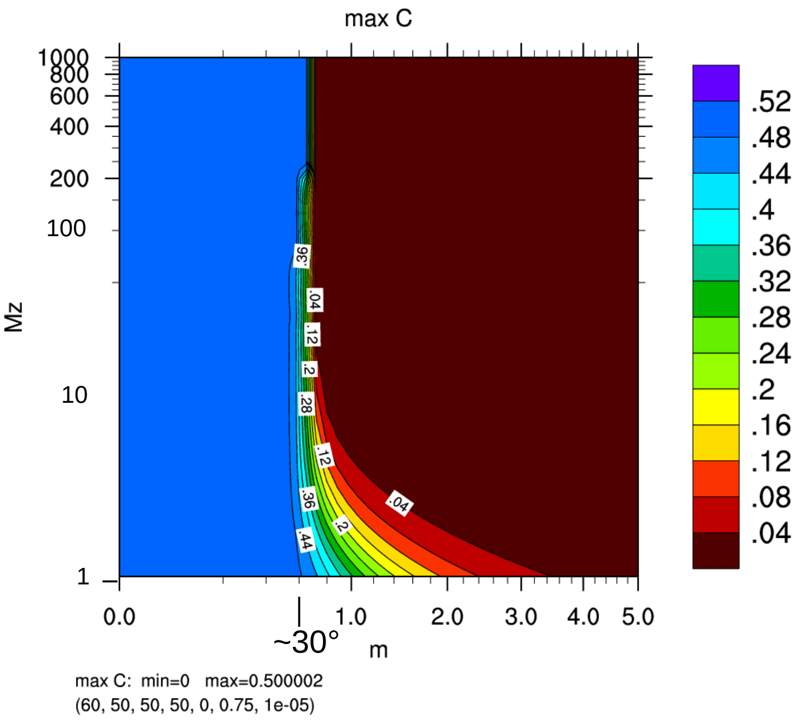
only 1D vertical diffusion implicit,
other terms explicit,
no off-centering (0.5)



Stability of 3D scalar diffusion in tilted terrain: max C_{Diff}

only 1D vertical diffusion implicit,
other terms explicit,
off-centering 0.75

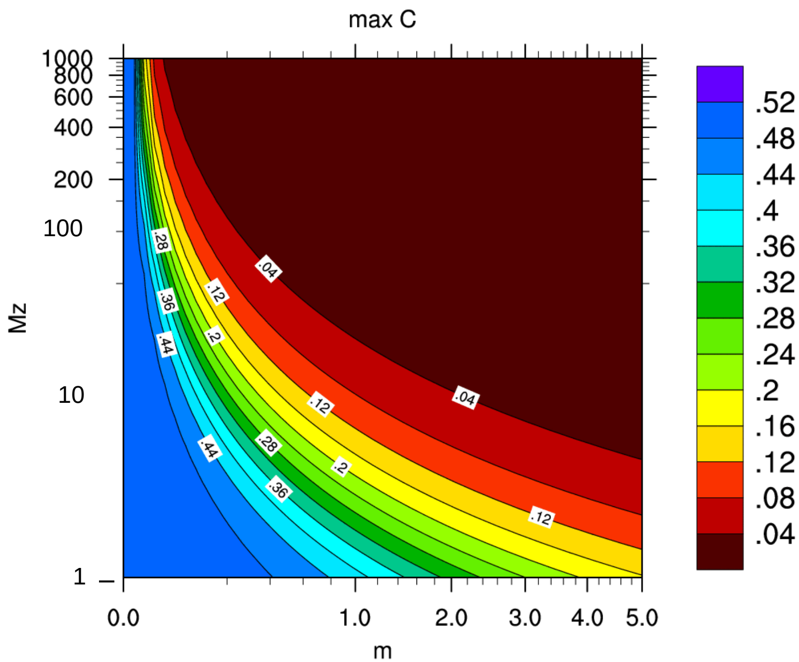
=old COSMO-version (until 5.2)



Stability of 3D scalar diffusion in tilted terrain: max C_{Diff}

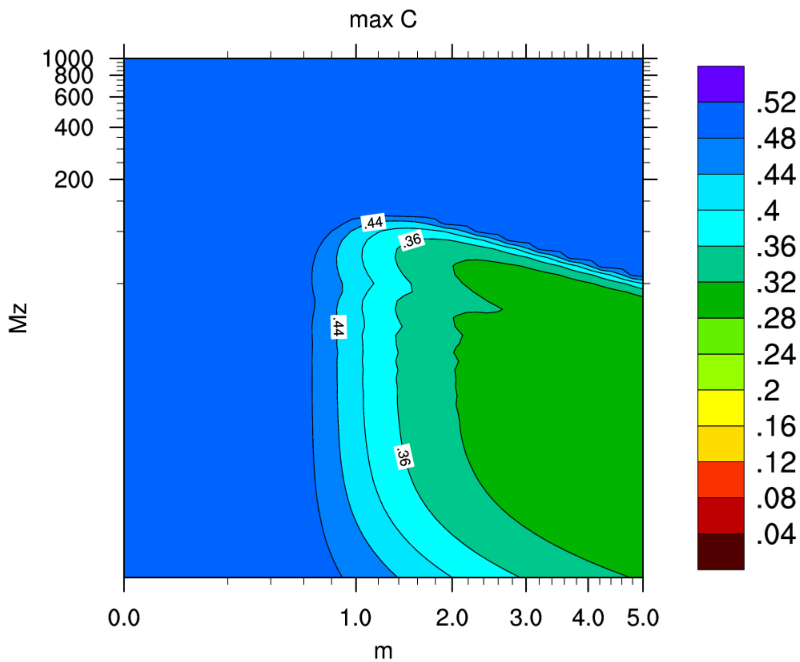
now: all terms with only vertical derivatives are treated implicitly

0.5 (=no off-centering)



max C: min=0.000205 max=0.500002
(60, 50, 50, 50, 0.5, 0.5, 1e-05)

off-centering 0.6

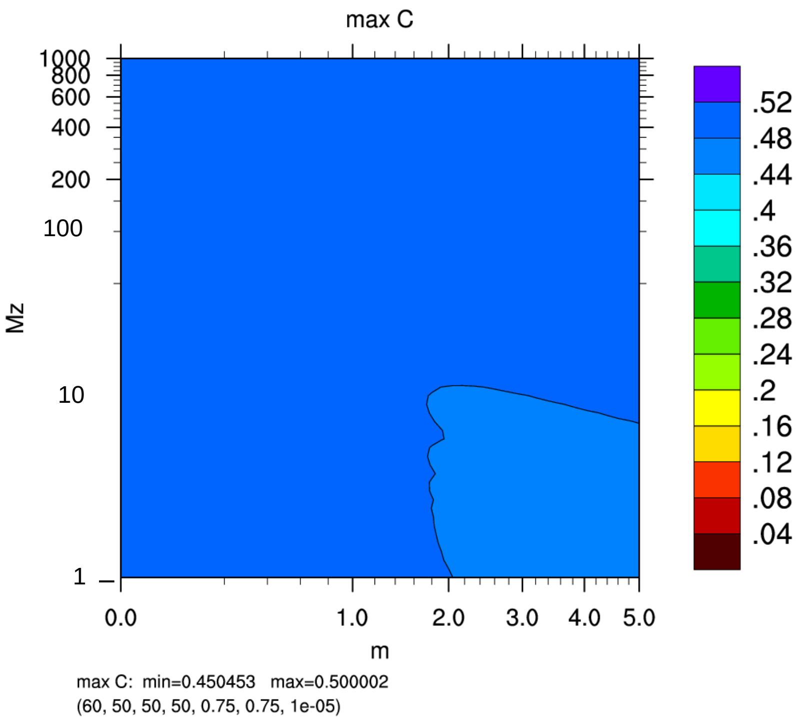


max C: min=0.285521 max=0.500002
(60, 50, 100, 100, 0.6, 0.6, 1e-05)

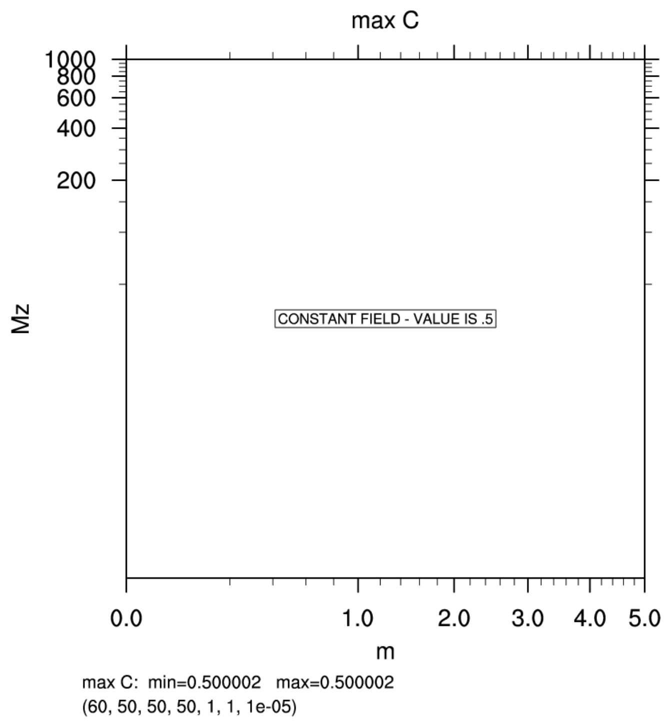
Stability of 3D scalar diffusion in tilted terrain: max C_{Diff}

now: all terms with only vertical derivatives are treated implicitly

off-centering 0.75



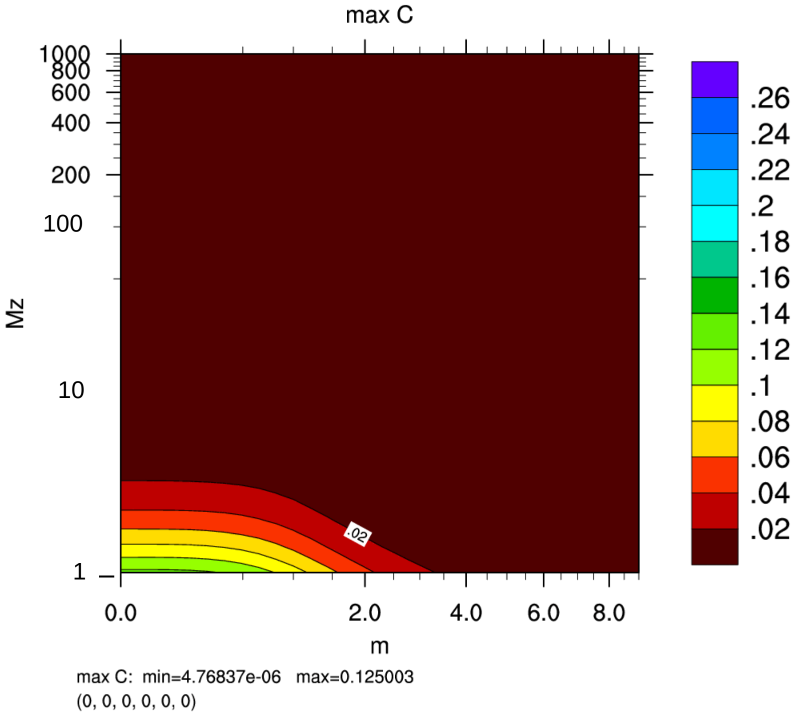
1.0 (=fully vertical implicit)



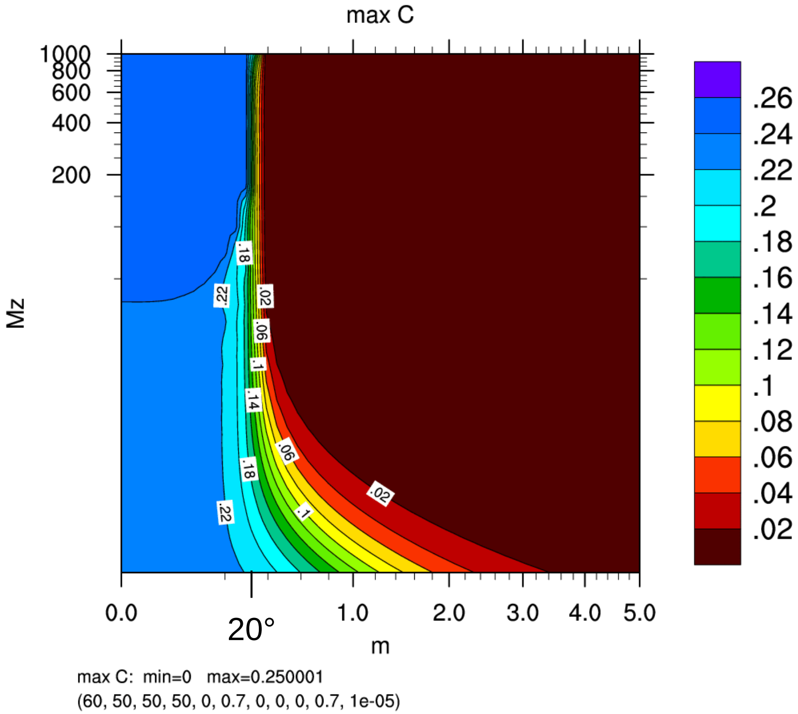
Numerical stability for vector diffusion

Stability of 3D vector diffusion in tilted terrain: max C_{Diff}

purely explicit



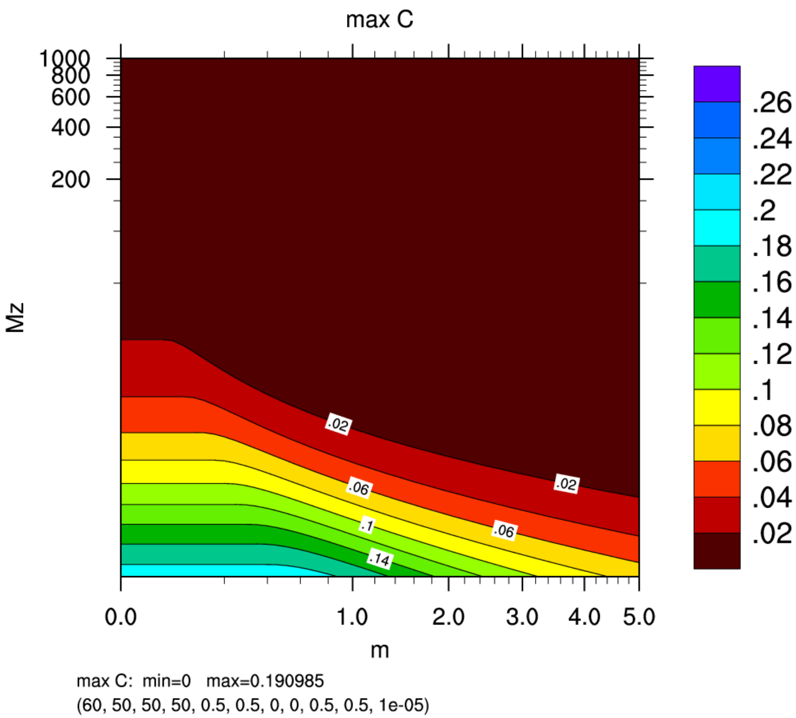
only 1D vertical diffusion implicit, other terms explicit, off-centering=0.7 (=old COSMO)



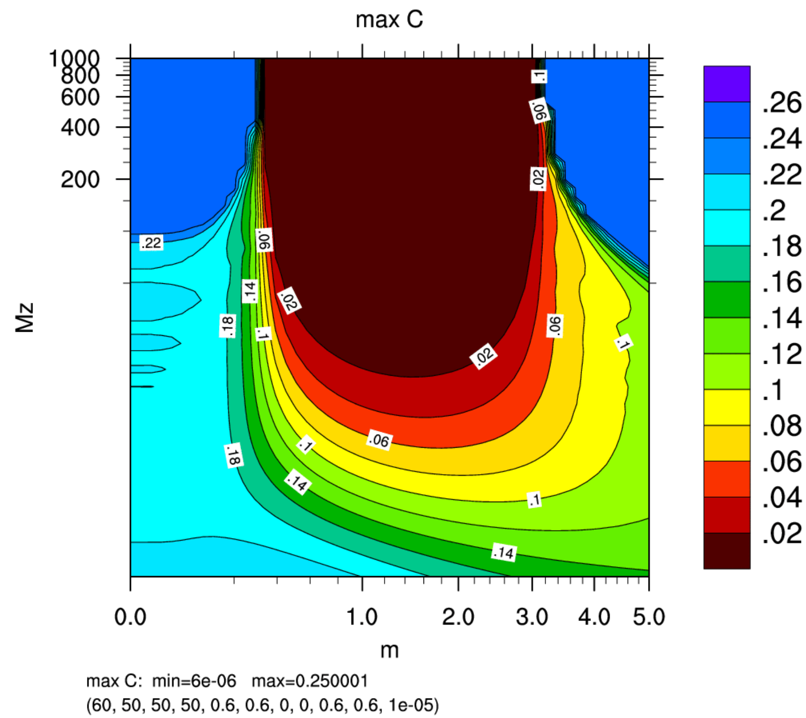
Stability of 3D <u>vector</u> diffusion in tilted terrain: max C_{Diff}

now: all terms with only vertical derivatives treated implicitly

no off-centering (0.5)



off-centering (0.6)

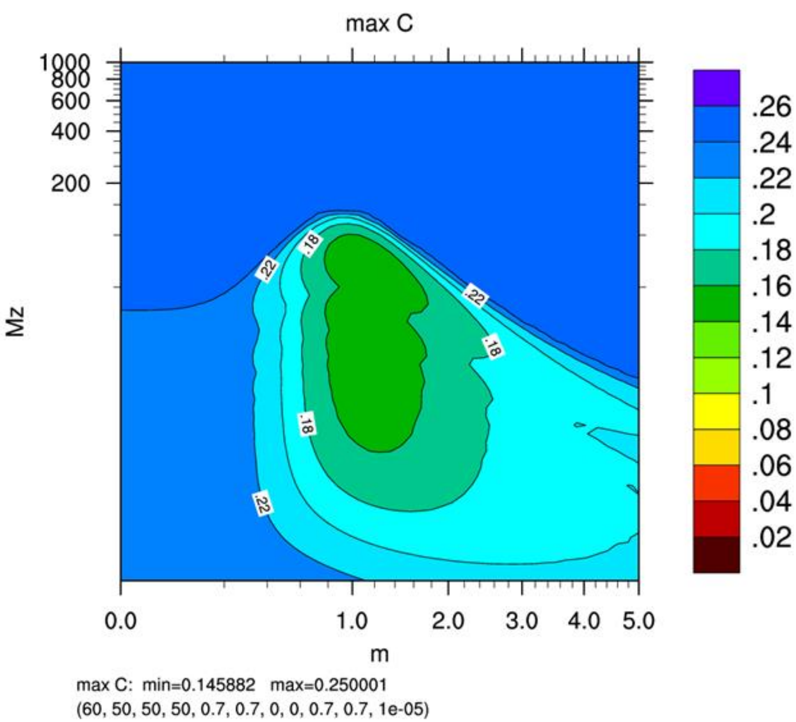


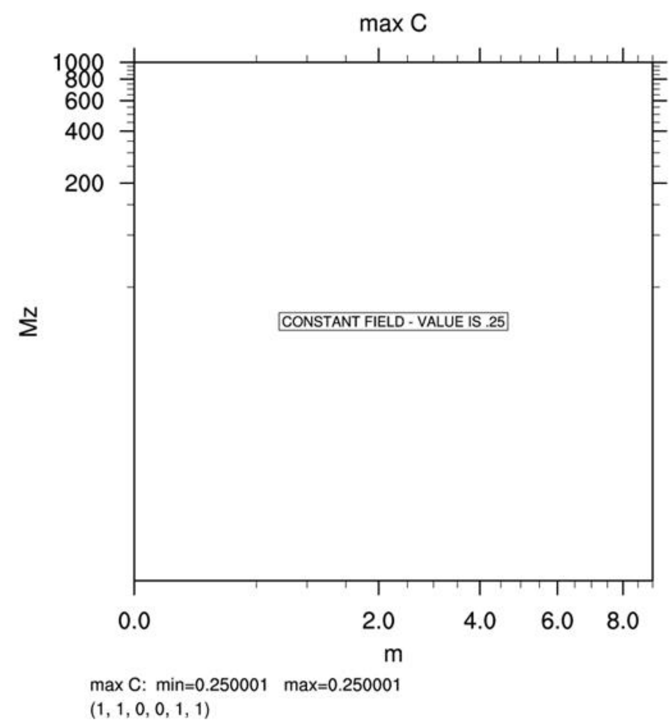
Stability of 3D <u>vector</u> diffusion in tilted terrain: max C_{Diff}

now: all terms with only vertical derivatives treated implicitly

off-centering (0.7)

vertically fully implicit (1.0)

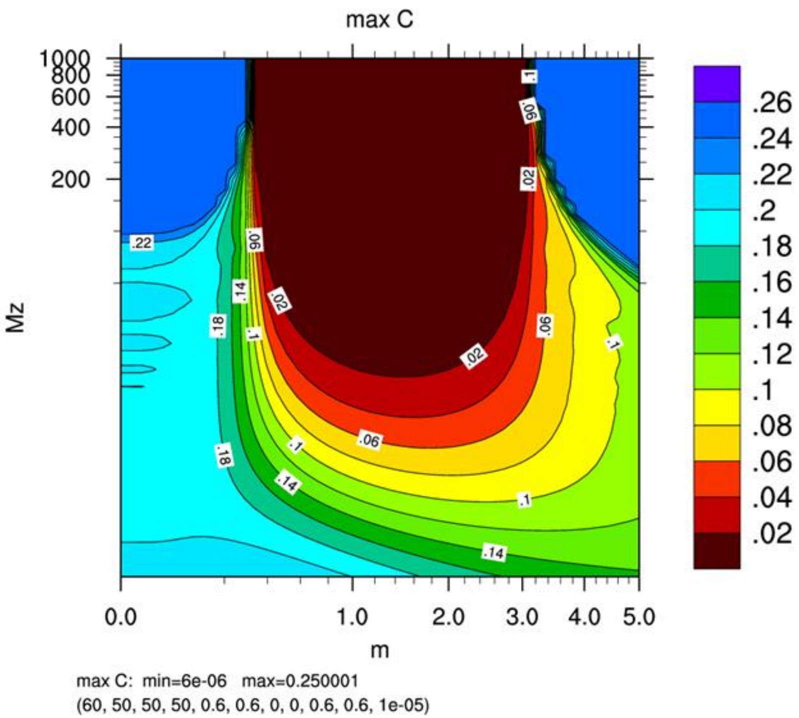




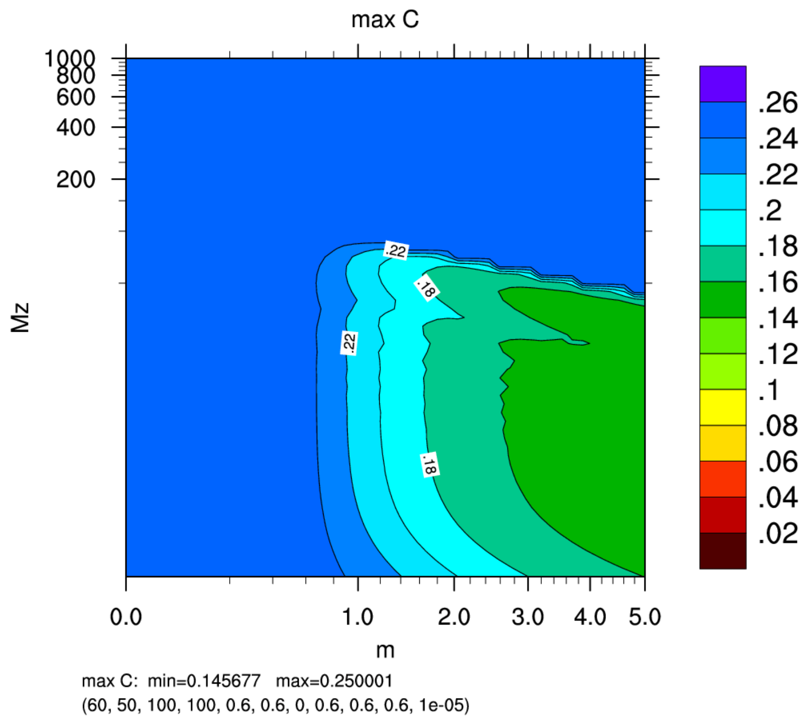
Stability of 3D <u>vector</u> diffusion in tilted terrain: max C_{Diff}

now: all terms with only vertical derivatives treated implicitly, off-centering 0.6

‚normal'



u,v forward, w backward



Testing

Test of scalar diffusion:
3-dim. isotropic gaussian tracer distribution

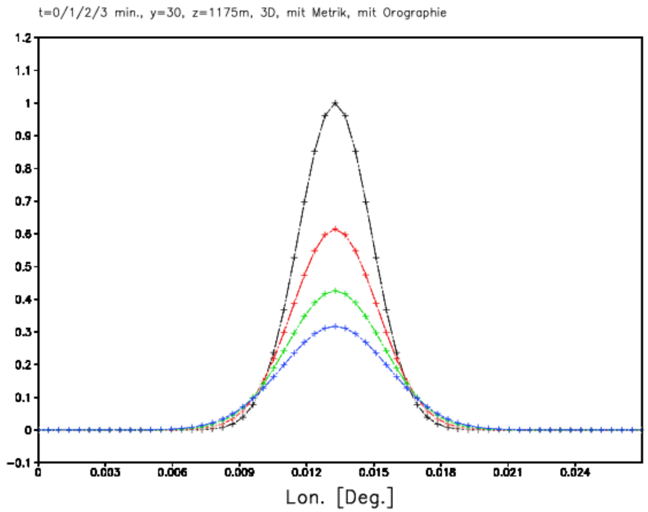
3D diffusion equation with $K=const.$:

$$\frac{\partial \phi}{\partial t} = K \Delta \phi$$

analytic Gaussian solution for K=const.:

$$\phi(r,t) = \frac{\Phi_0}{\sqrt{4\pi K (t+t_0)^3}} \exp\left(-\frac{r^2}{4K(t+t_0)}\right),$$

$$r := \sqrt{x^2+y^2+z^2}$$



Baldauf (2005) COSMO-Newsl. no. 6

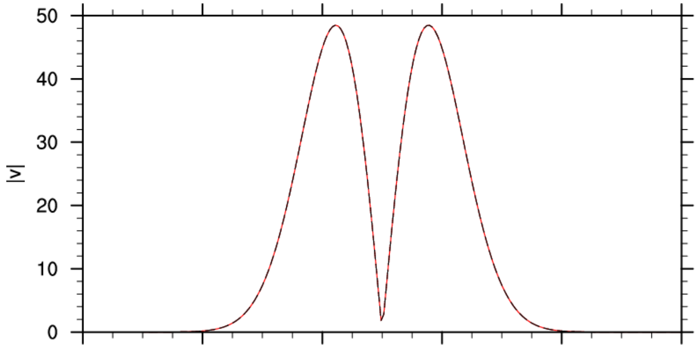
A proposal for an analogous test case for 3D vector diffusion:

isotropic, purely radial vector field:

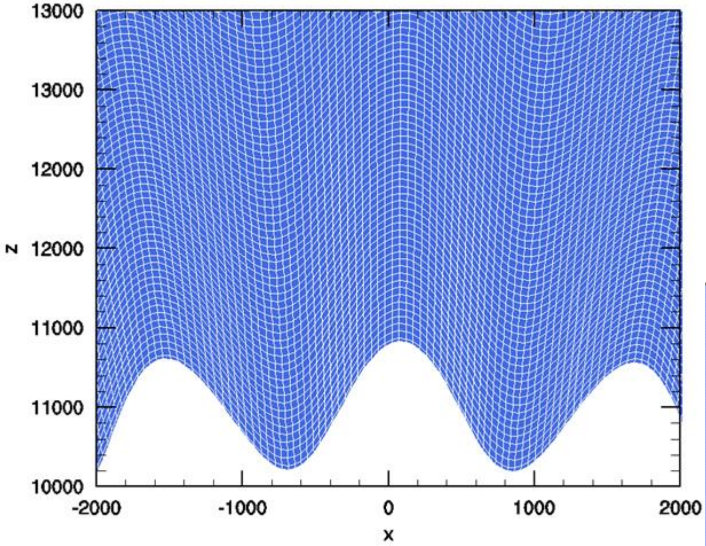
$$\mathbf{v}(\mathbf{r}, t) = v_r(r, t)\hat{\mathbf{e}}_r$$

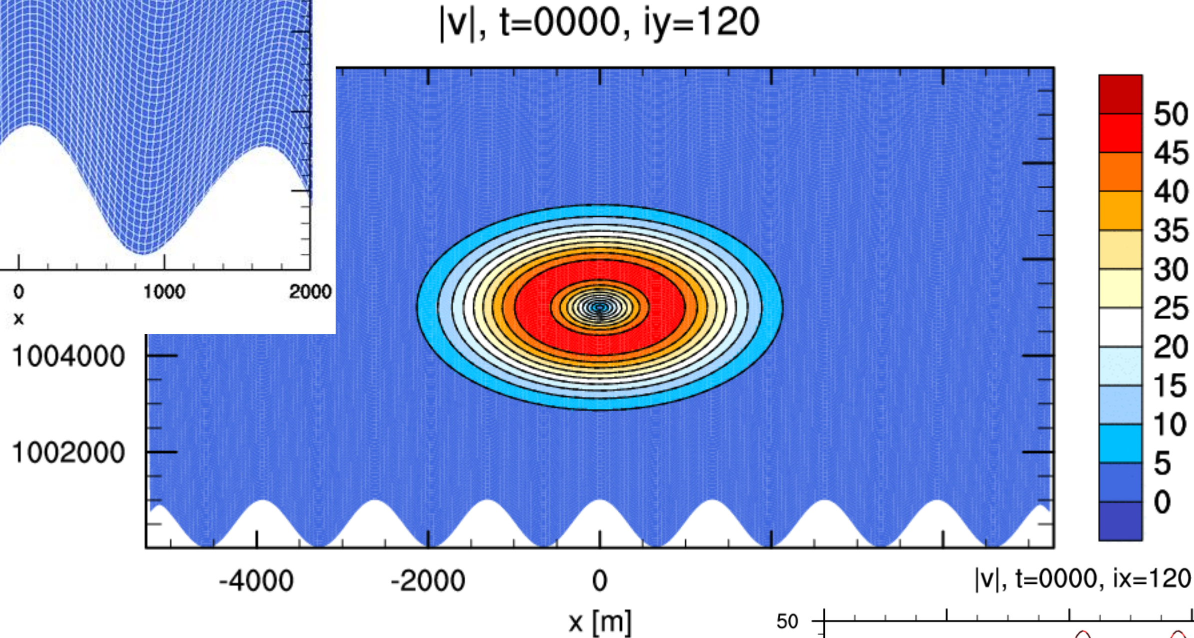
with

$$v_r(r, t) = const. \frac{r}{\sqrt{K}(t + t_0)^{5/2}} \cdot e^{-\frac{r^2}{8K(t+t_0)}}$$



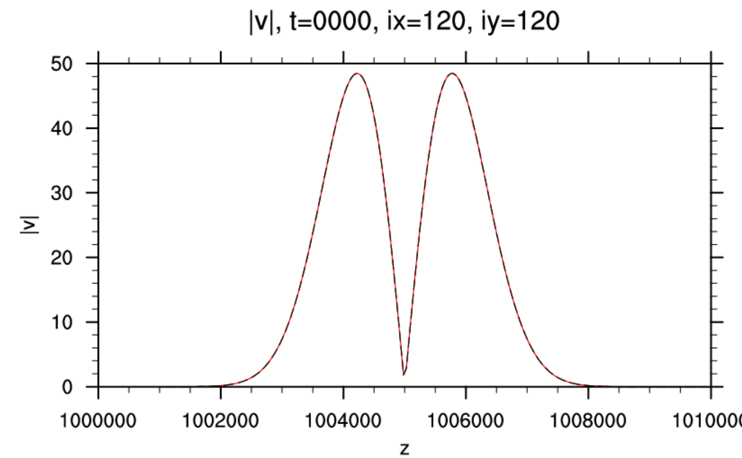
Vector diffusion test





v,analy: min=0 max=48.5371
v,simul: min=0 max=48.5371

5.1r39_50_v_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrF

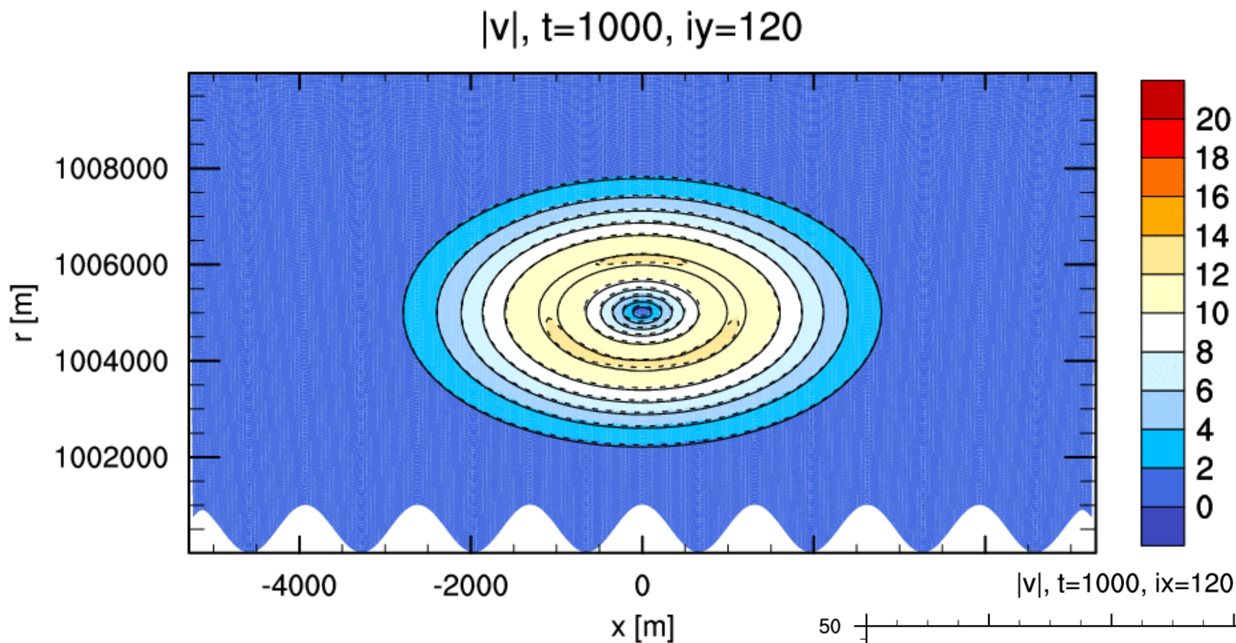


v,analy: min=5.64219e-07 max=48.4769
v,simul: min=5.64219e-07 max=48.4769

5.1r39_50_v_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrF

analytic solution: solid lines
COSMO solution: colors + dashed lines

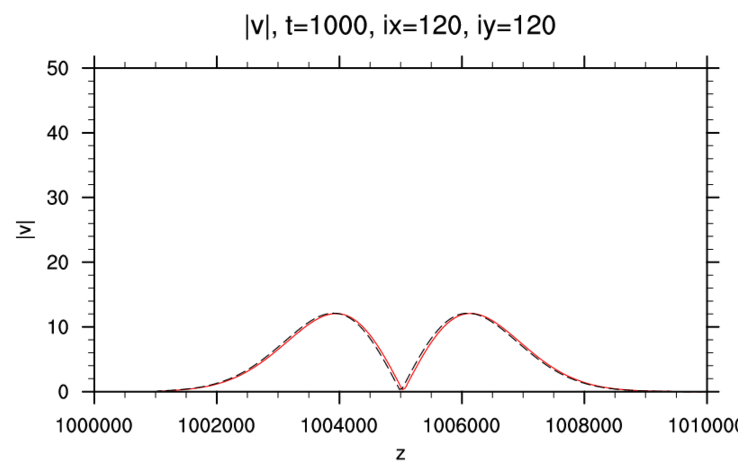
Vector diffusion test



v,analy: min=0 max=12.135

v,simul: min=0 max=12.1405

5.1r39_50_v_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrF



v,analy: min=0.0029977 max=12.1255

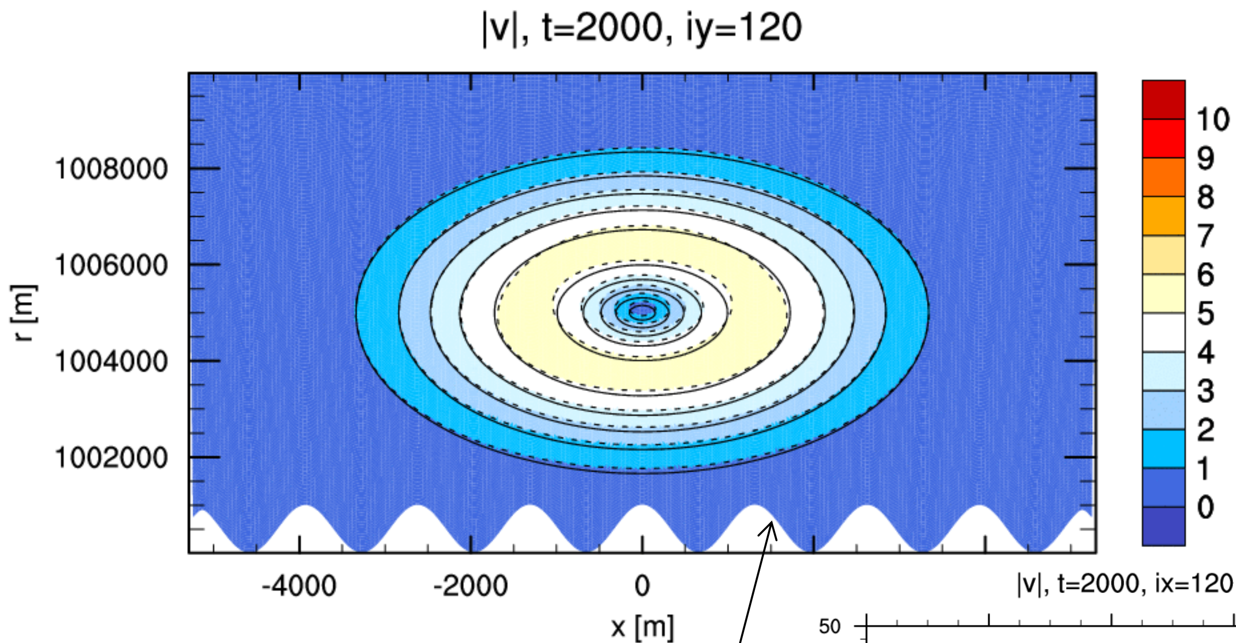
v,simul: min=0.000584523 max=12.087

5.1r39_50_v_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrF

analytic solution: solid lines

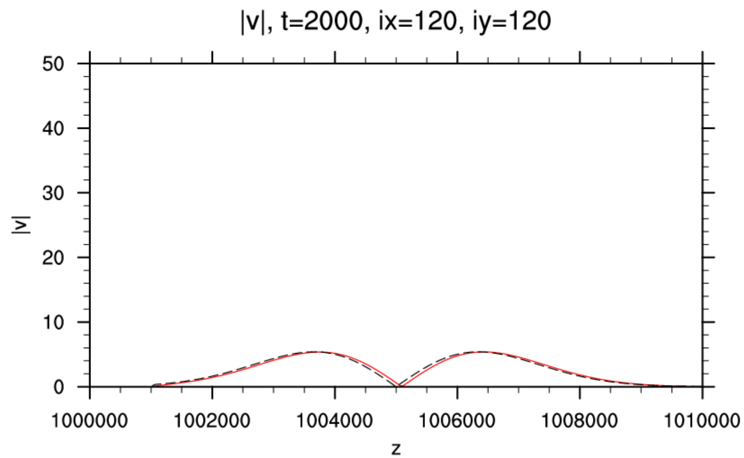
COSMO solution: colors + dashed lines

Vector diffusion test



v,analy: min=0 max=5.3929
v,simul: min=0 max=5.41991

5.1r39_50_v_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_ImetrF



v,analy: min=0.0338866 max=5.38961
v,simul: min=0.00462753 max=5.3754

5.1r39_50_v_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_ImetrF

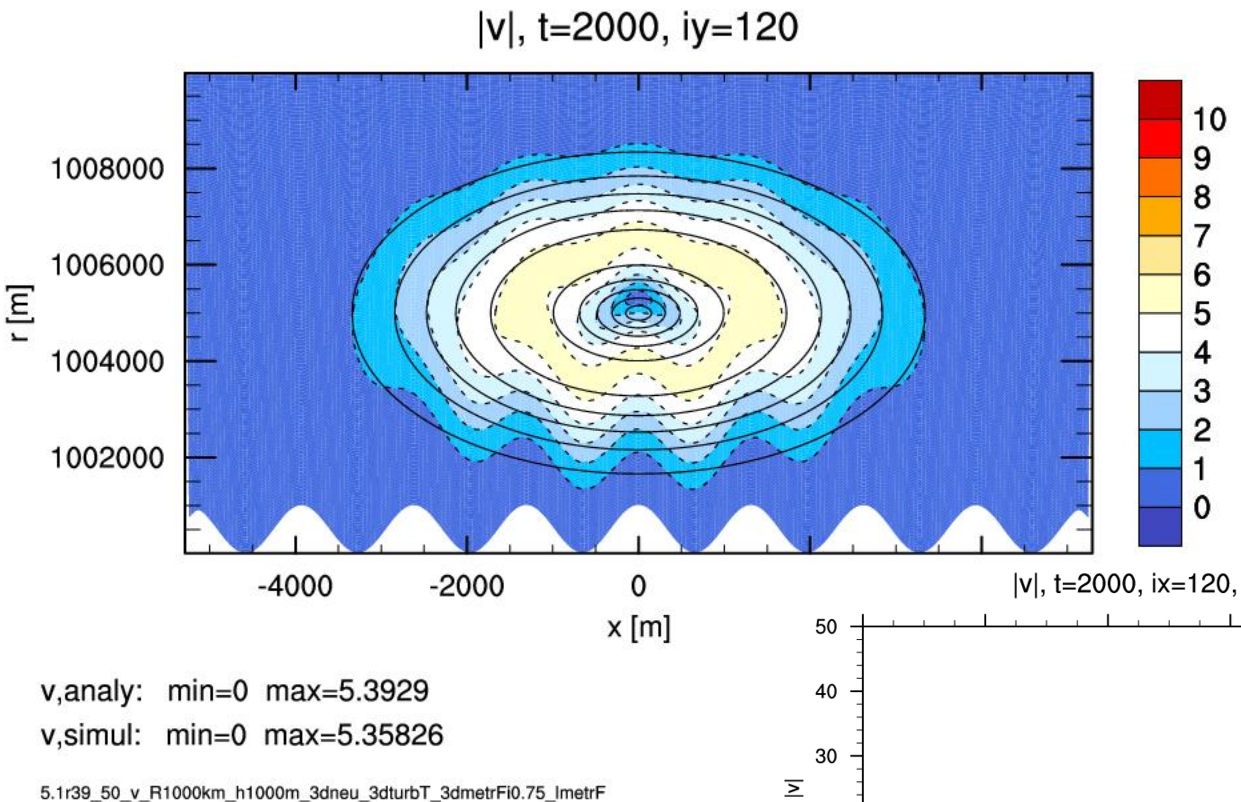
analytic solution: solid lines
COSMO solution: colors + dashed lines

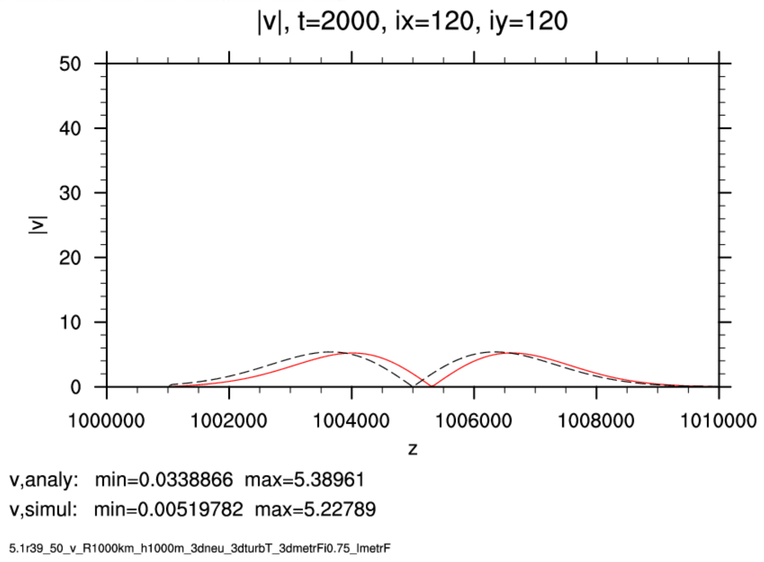
max slope ~ 2.3 ~ 67°

the run with only one vertical implicit term
(=old COSMO version) became unstable!

for comparison:
without metric terms

Vector diffusion test





analytic solution: solid lines
COSMO solution: colors + dashed lines

Until now, only the pure 3D diffusion was considered.

Physics-dynamics coupling in COSMO:
physics tendencies are added to the slow dynamic processes and treated in the 3-stage Runge-Kutta time split scheme (*Wicker, Skamarock, 2002, MWR*)

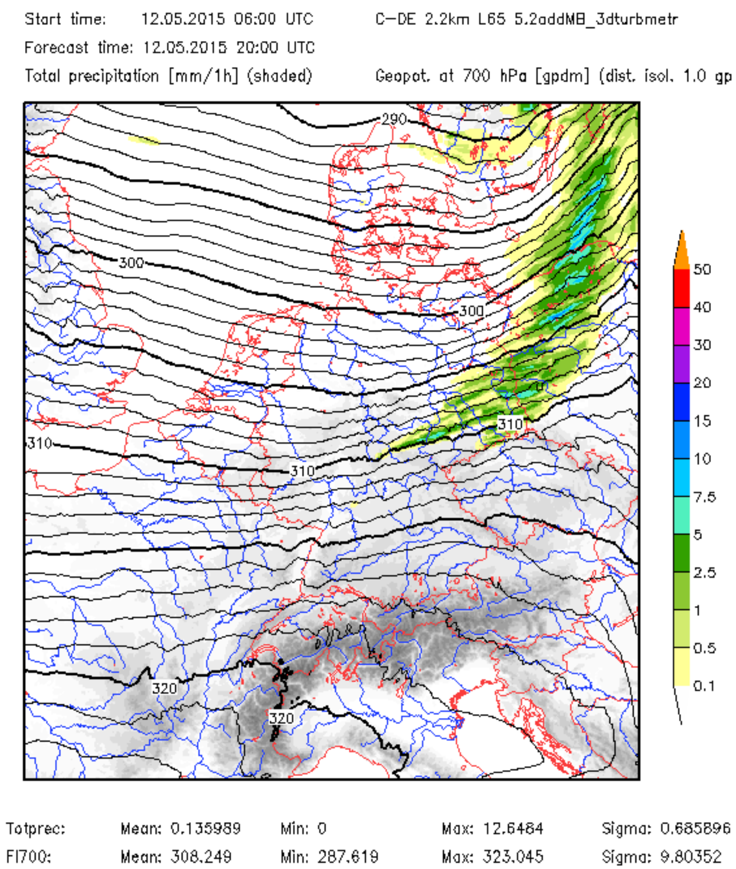
Real test case

12 May 2015, 06 UTC run
at this day COSMO-DE (2.8 km, L50) missed several convective events which produced heavy rain and intensive gusts.

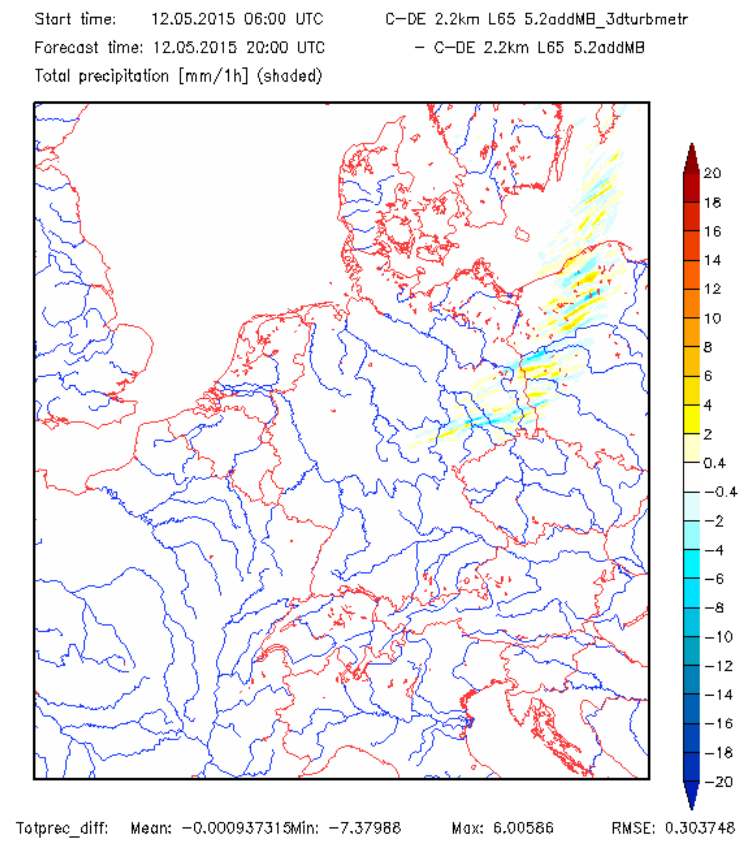
Shown are results of COSMO-D2 (2.2 km, L65) at 20 UTC (i.e. after 14h forecast time)

Real case: ‚12 May 2015, 06 UTC run‘, COSMO-D2, 1h precipitation sum

with 3D diffusion

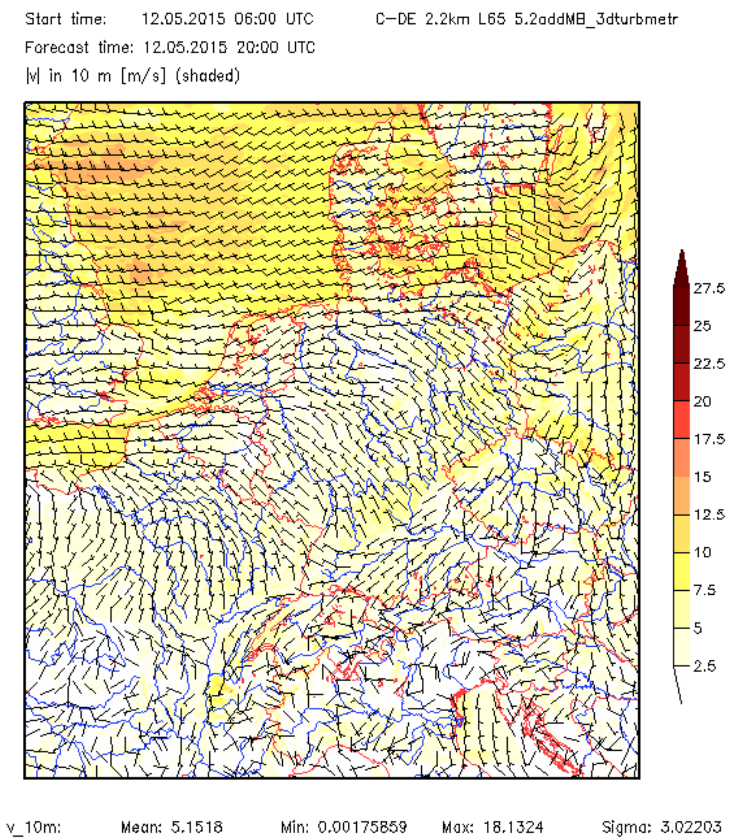


difference to 1D diffusion

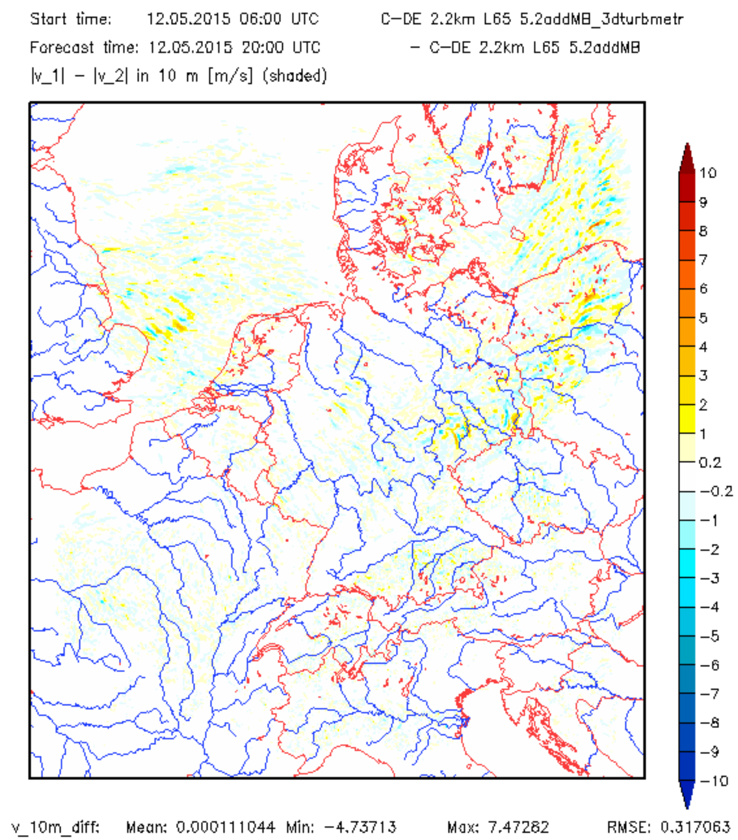


Real case: ‚12 May 2015, 06 UTC run‘, COSMO-D2, 10m wind

with 3D diffusion



difference to 1D diffusion



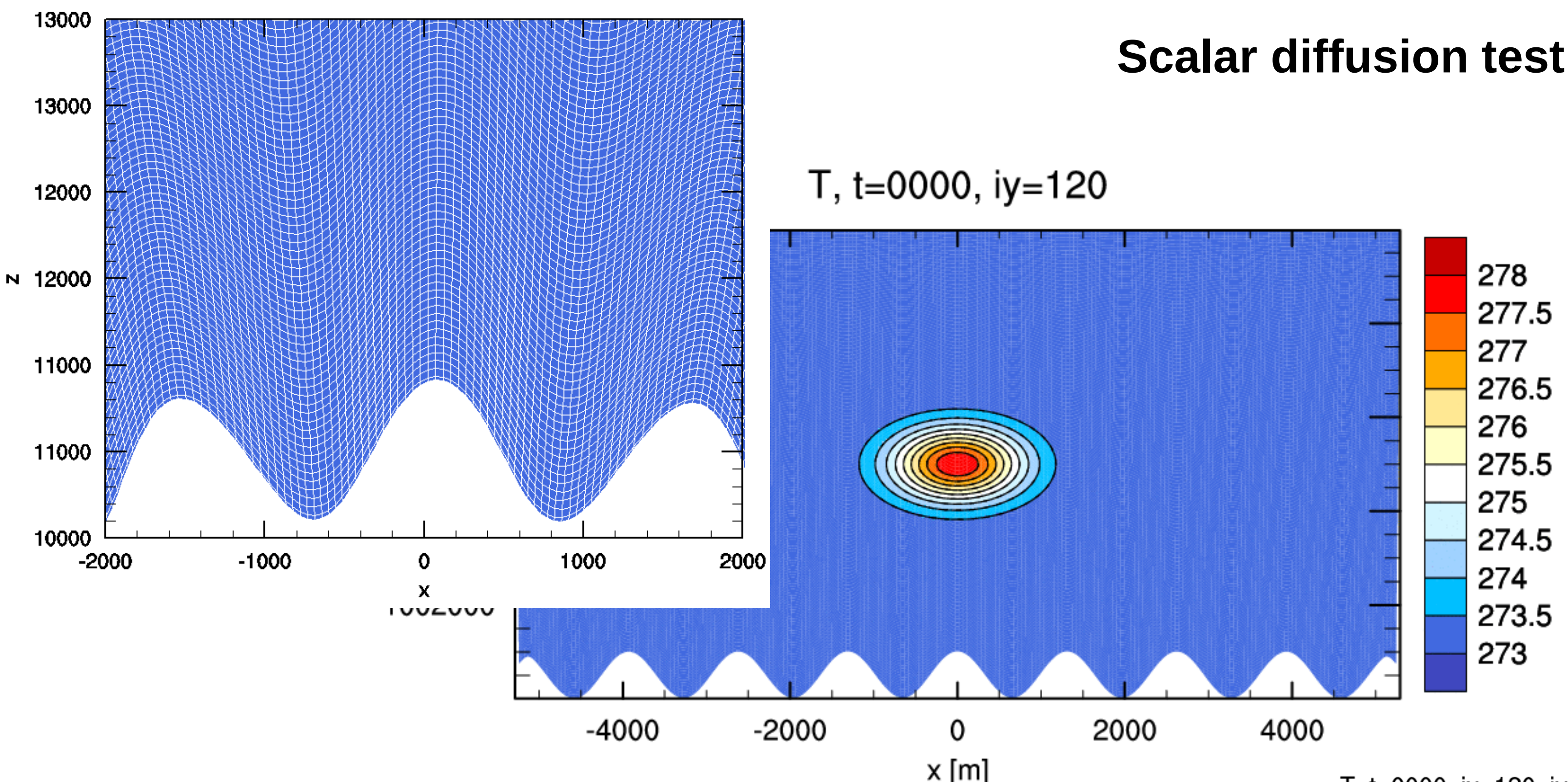
Summary

- Stability analysis indicates that 3D diffusion in terrain following coordinates and HE-VI approach may be stable in *arbitrary steep* terrain if
  - use all terms with only z-deriv. in the tridiagonal solver
  - some off-centering (~0.7) is necessary and leads to

$$K \Delta t \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \leq$$ 0.45 (scalar case)
0.146 (vector case)

  - in the v-equations additional forward-backward even slightly increases stability (not implemented/planned yet)
- Testing by idealised tests with known analytic solution successfully carried out both for scalar diffusion (*Baldauf, 2005*) and by a new vector diffusion test
- New implementation runs stable in real case simulations
- Available in COSMO version 5.3
- Will be soon available in ICON (work by *Slavko Brdar*)
- Publication *Baldauf, Brdar* (accepted by *QJRMS* after minor corr.)

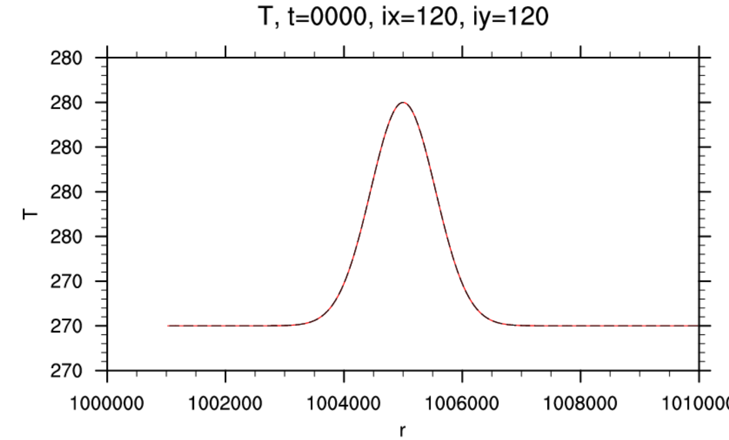
Scalar diffusion test



T,analy: min=273 max=277.997

T,simul: min=273 max=277.997

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrT



T,analy: min=273 max=277.997

T,simul: min=273 max=277.997

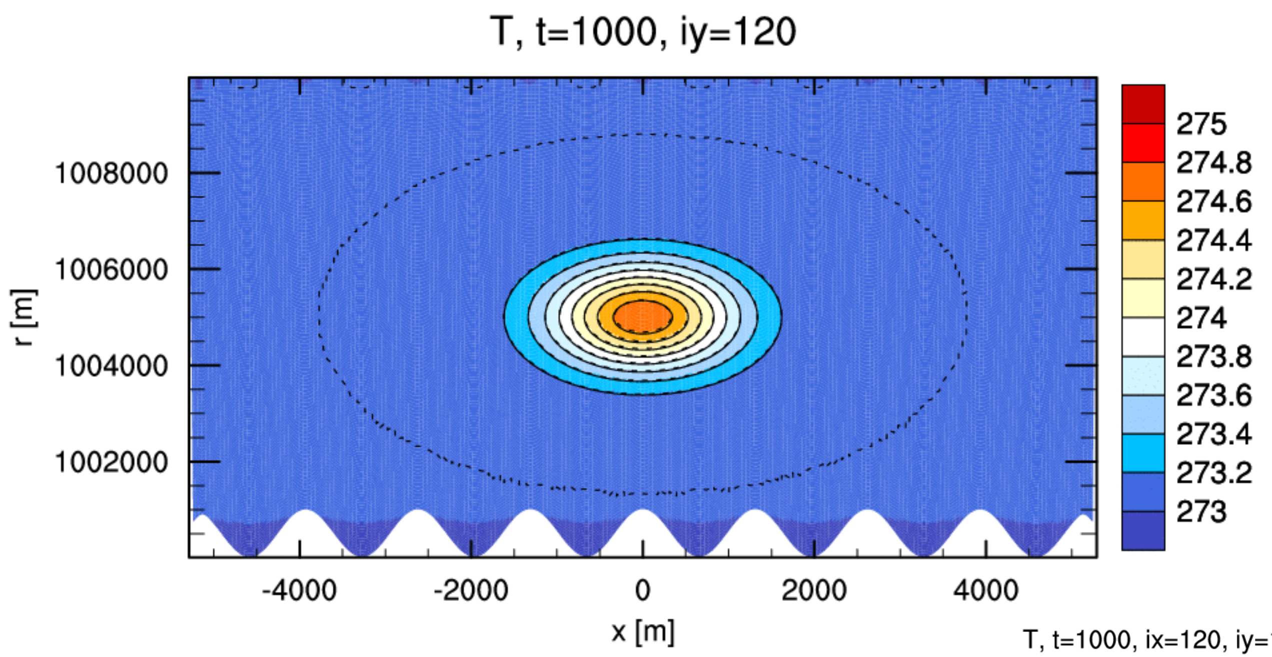
5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrT

analytic solution: solid lines

COSMO solution: colors + dashed lines

R=1000km ('real' planet)

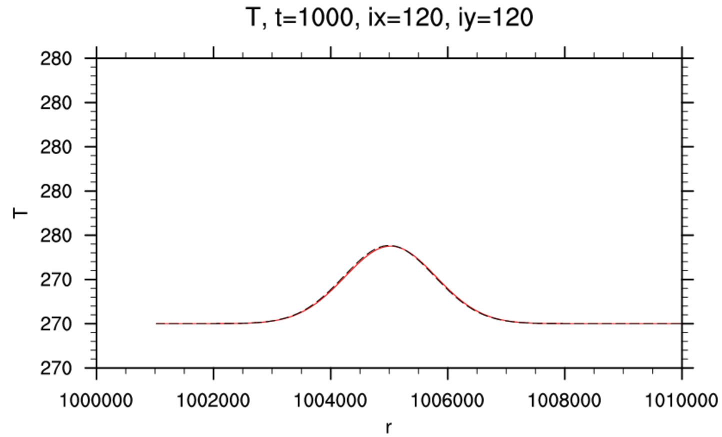
Scalar diffusion test



T,analy: min=273 max=274.767

T,simul: min=273 max=274.753

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrT



T,analy: min=273 max=274.767

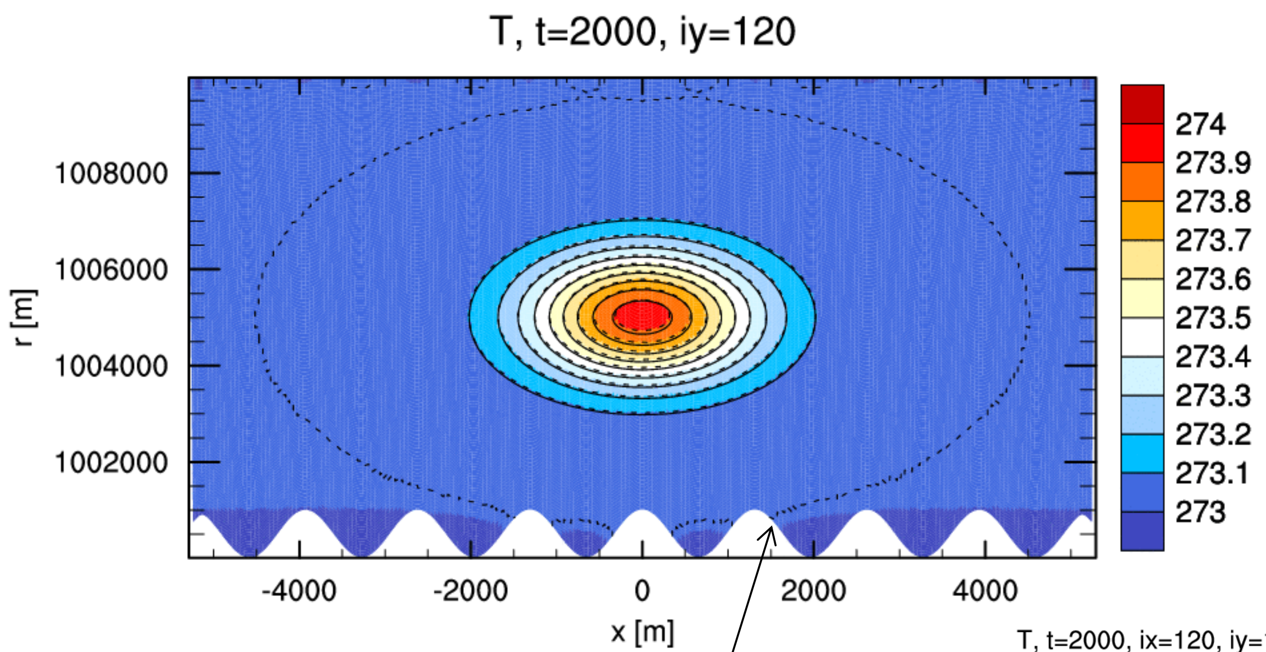
T,simul: min=273 max=274.753

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrT

analytic solution: solid lines

COSMO solution: colors + dashed lines

Scalar diffusion test



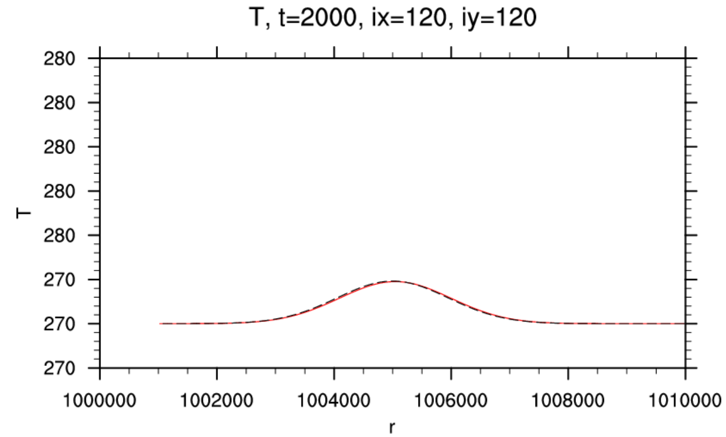
T,analy: min=273 max=273.962

T,simul: min=273 max=273.952

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrT

max slope ~ 2.3 ~ 67°

the run with only one vertical implicit term (=old COSMO version) became unstable!

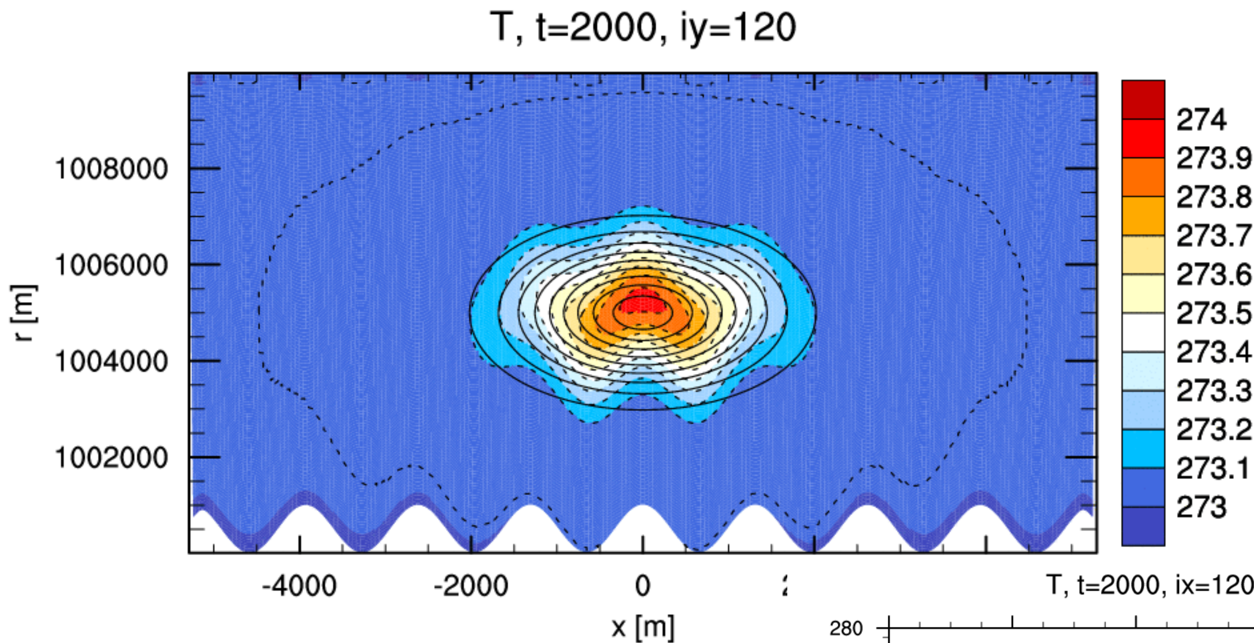


T,analy: min=273 max=273.962

T,simul: min=273 max=273.952

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrTi0.75_lmetrT

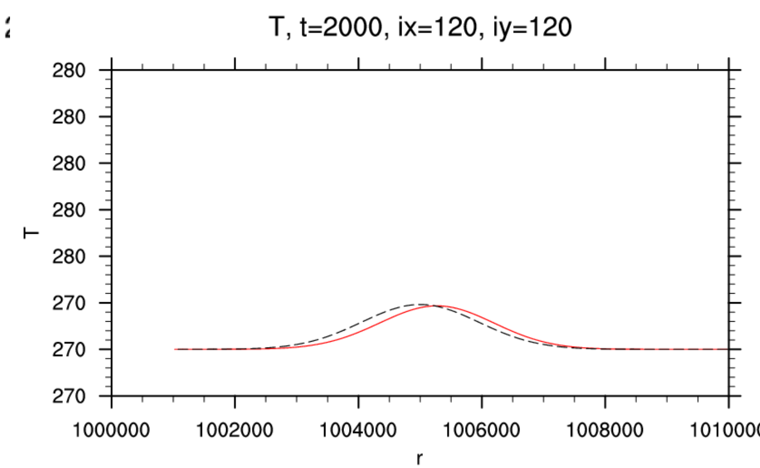
Scalar diffusion test



T,analy: min=273 max=273.962

T,simul: min=273 max=273.934

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrFi0.75_lmetrT



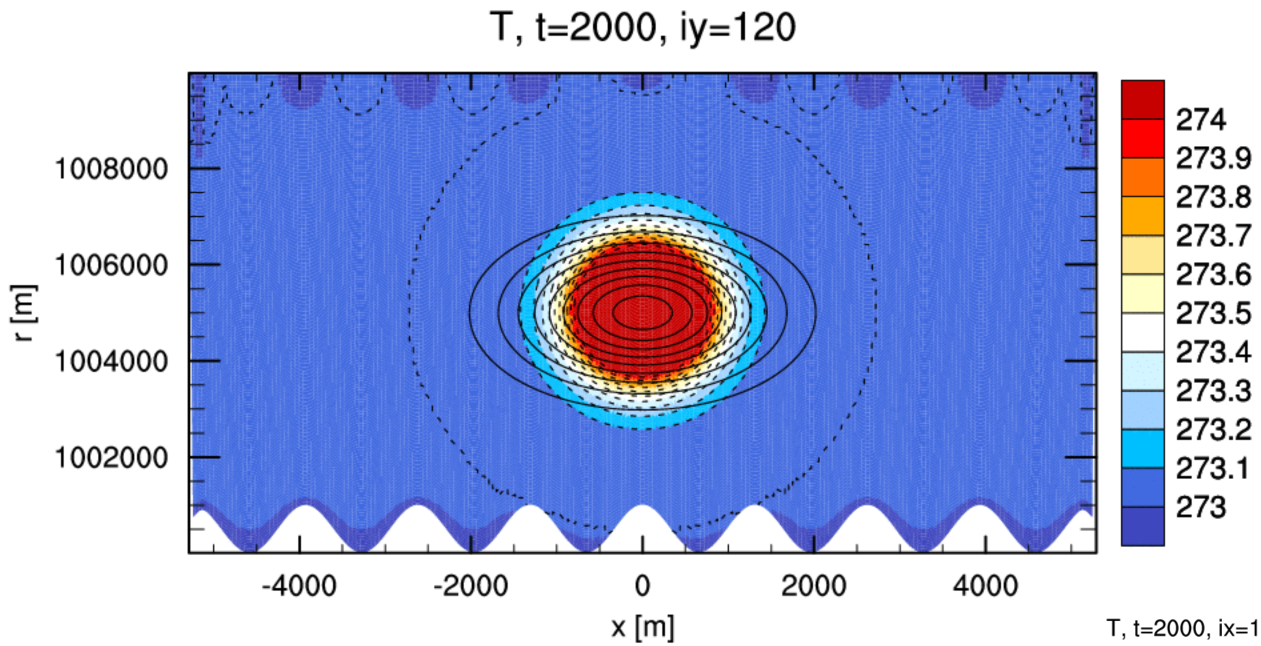
T,analy: min=273 max=273.962

T,simul: min=273 max=273.934

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbT_3dmetrFi0.75_lmetrT

for comparison:
without metric terms

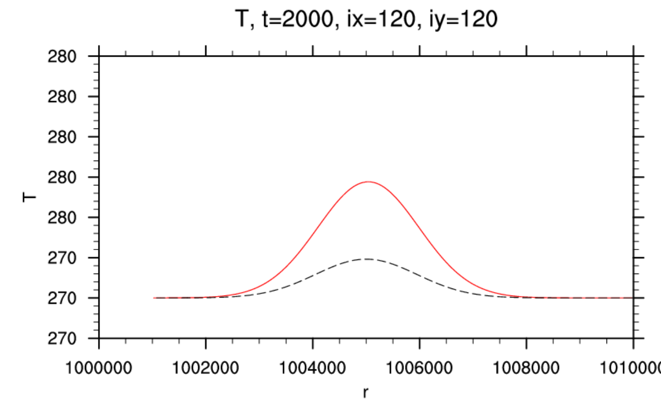
Scalar diffusion test



T,analy: min=273 max=273.962

T,simul: min=272.999 max=275.882

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbF_3dmetrFi0.75_lmetrT



T,analy: min=273 max=273.962

T,simul: min=273 max=275.882

5.1r39_50_s_R1000km_h1000m_3dneu_3dturbF_3dmetrFi0.75_lmetrT

only 1D (vertical) diffusion